

# Questions on Quantization

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ABSTRACT. A number of basic questions concerning quantization within the setting of operator algebras are stated, and in the process a brief survey of some of the recent developments is given.

I collect here a number of basic questions concerning quantization within the setting of operator algebras, while at the same time surveying briefly some of the recent developments. For a less up-to-date but more leisurely survey see [R8]. Here I will only give reference to relatively recent papers. In particular, I will not repeat many important references which already appear in [R8]. An extensive treatment of many of the topics mentioned here, including historical notes and references, will appear in a forthcoming book by Landsman [La3]. See [Wr] for a recent comparison of various approaches to quantization, going beyond the operator algebra setting. For a fine survey of the approach using coherent states (which has significant interaction with operator algebras) see [AG]. For an interesting collection of questions about quantum groups in the operator algebra setting see [Wa].

The present manuscript is based on informal notes (based on earlier talks which I gave) which I distributed at the Dartmouth Workshop on E-theory, Quantization, and Deformations, September 1997. A number of people subsequently gave me suggestions for improvements, which are incorporated here. I am especially grateful to N. Landsman and A. Weinstein for their suggestions. I would, of course, very much like to be told about any progress on answering any of the questions I ask here.

The kinematics of a classical mechanical system is given by its phase space, which is a manifold, say  $M$ , equipped with a Poisson bracket. It should be the “shadow” of a quantum mechanical system, whose kinematics is given by specifying a non-commutative  $C^*$ -algebra of operators and suitable “physical” states on it. There should be a length-scale,  $\hbar$ , Planck’s constant. As  $\hbar \rightarrow 0$  the quantum system should converge in some sense to the classical system. We will not deal here with dynamics.

One natural approach to finding quantum systems of which a classical system is the “shadow” is to deform the pointwise product on the algebra of functions on  $M$  into a family, parametrized by  $\hbar$ , of non-commutative products. There is an extensive literature dealing with the purely algebraic side of this, in which the

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product is a formal power-series in  $\hbar$  whose coefficients are functions (and thus the product of two functions is not again a function). See [W1] and the references therein. But formal power-series do not mesh well with operator algebras, and so we will not discuss this further here.

We look for deformed products which yield actual functions. We will work with bounded operators, so we want bounded functions. The algebra  $C_\infty(M)$  of continuous complex-valued functions on  $M$  which vanish at infinity determines  $M$  as its maximal ideal space. The Poisson bracket is only defined on functions which are differentiable in suitable directions, so we work with dense subalgebras of  $C_\infty(M)$ . We want  $C^*$ -algebras, so we must allow also deformation of the involution (complex conjugation) and of the  $C^*$ -norm (the supremum norm) on  $C_\infty(M)$ .

**1. Definition [R5, R8].** Let  $M$  be a Poisson manifold. By a *strict deformation quantization* of  $M$  we mean a dense  $*$ -subalgebra  $A$  of  $C_\infty(M)$  which is carried into itself by the Poisson bracket, together with a closed subset  $I$  of the real line containing 0 as a non-isolated point, and for each  $\hbar \in I$  a product  $\times_\hbar$ , an involution  ${}^*\hbar$ , and a  $C^*$ -norm  $\| \cdot \|_\hbar$  on the linear space of  $A$ , such that for  $\hbar = 0$  they are the product, involution, and norm from  $C_\infty(M)$ , and such that

1. The completions of  $A$  for the various  $C^*$ -norms form a continuous field of  $C^*$ -algebras over  $I$ .
2. For  $f, g \in A$  we have

$$\|(f \times_\hbar g - g \times_\hbar f)/\hbar - i\{f, g\}\|_\hbar \rightarrow 0$$

as  $\hbar \rightarrow 0$ .

We call attention to a further aspect which has not received much emphasis previously.

**2. Definition.** We will say that a strict deformation quantization is *flabby* if  $A$ , as above, contains  $C_c^\infty(M)$ , the algebra of smooth functions of compact support on  $M$ .

The most important present source of flabby strict deformation quantizations is the Weyl–Moyal quantization and its generalizations. One fairly far-reaching generalization goes as follows [R5]. Let  $\alpha$  be an action of  $V = \mathbb{R}^n$  on a locally compact space  $M$ , and so on  $C_\infty(M)$ . Let  $A$  be the dense  $*$ -subalgebra of smooth vectors in  $C_\infty(M)$  for  $\alpha$ . Let  $X_j$  denote the derivation of  $A$  in the  $j$ -th direction of  $\mathbb{R}^n$  via  $\alpha$ . The choice of a skew-symmetric  $n \times n$  matrix  $J$  determines a Poisson bracket on  $A$  defined by

$$\{f, g\} = \sum J_{jk}(X_j f)(X_k g).$$

For each  $\hbar \in \mathbb{R}$  define a new product,  $\times_\hbar$ , on  $A$  by

$$f \times_\hbar g = \int_{V \times V} \alpha_{\hbar Ju}(f) \alpha_v(g) e^{2\pi i u \cdot v} du dv,$$

where this integral must be interpreted as an oscillatory integral [R5]. Let the involution remain undeformed. There is a natural way to define  $C^*$ -norms  $\| \cdot \|_\hbar$ , discussed in [R5]. This provides a flabby strict deformation quantization of  $M$  with the given Poisson bracket.

Notice that the above is a **universal** deformation formula, in the sense that it works any time  $\mathbb{R}^n$  acts on *any* space or  $C^*$ -algebra.

**3. Question.** Are there universal deformation formulas, at the analytical level, for actions of (at least some) other Lie groups?

At the Lie algebra level and in terms of formal power series, a few such universal deformation formulas are known. See [GZ] and references therein. In particular, there is a universal deformation formula for the Lie algebra of the affine group of the real line (and a related formula appears in slightly buried form in [Oh]). This suggests that at least for actions of the affine group there should be a universal deformation formula at the analytical level.

For actions of  $\mathbb{R}^n$  many specific examples of quantum spaces constructed as strict deformation quantizations are given in [R5, R8], such as quantum disks, tori and spheres. Quantum groups can be constructed by this method [R6, R9, Ln]. Closely related constructions provide quantum Heisenberg manifolds and lens spaces [R1, Ab, AE1, AE2], as well as an algebra for the space-time uncertainty relations [R10]. (Probably the closely related constructions in [LR1, LR2] can be arranged, in the appropriate situations, to give flabby strict deformation quantizations, though I have not checked this.) But it is unclear to me how often the above construction applies, because I have found little information about:

**4. Question.** Given a Poisson bracket, how does one determine whether it comes from an action of  $\mathbb{R}^n$  as described above, for some  $n$ ? In particular, what cohomological obstructions are there to expressing a Poisson bracket globally in terms of a family of *commuting* vector fields?

In answer to my query following a suggestion of Alan Weinstein, Charles Pugh very recently showed me a proof that any action of  $\mathbb{R}^n$  on the 2-sphere, for any  $n$ , must have a fixed point. This shows that no Poisson bracket on the 2-sphere which comes from a symplectic structure can be given by an action of  $\mathbb{R}^n$ . (But the question of whether there exists some strict deformation quantization of the 2-sphere for some symplectic structure remains open.)

As a particularly nice situation in which one can ask Question 4 we have:

**5. Question.** Let  $M$  be a manifold with a Poisson bracket coming from an action of  $\mathbb{R}^n$  as above. Let  $\alpha$  be a free and proper action of a group  $G$  on  $M$  which preserves the Poisson bracket, so that one obtains a Poisson bracket on the quotient manifold  $M/\alpha$ . When does this latter Poisson bracket come from an action of  $\mathbb{R}^n$  on  $M/\alpha$ ? Also, the same question except for  $\alpha$  replaced by a more general suitable equivalence relation on  $M$ ?

A particular case of the process just described occurs in the construction of the quantum Heisenberg manifolds discussed in [R1, Ab, AE1, AE2]. For that case it is shown at the end of [AE2] that the Poisson bracket does in fact come from an action of  $\mathbb{R}^2$ . But it is not clear to me why this should be true in general.

**6. Question.** Given a Poisson manifold  $M$ , how does one determine whether there is a Poisson manifold  $N$  whose Poisson bracket comes from an action of  $\mathbb{R}^n$ , and a group action  $\alpha$  (or more general equivalence relation) on  $N$  preserving the Poisson bracket, such that  $N/\alpha \cong M$  as Poisson manifolds?

The reason for asking this question is that even when the answer to Question 5 is negative, one can hope to construct a deformation quantization of the quotient manifold from one of the big manifold:

**7. Question.** Given a Poisson manifold and a quantization of it, and nice actions of a group on both the manifold and the quantization which are compatible, how does one construct from this a quantization of the quotient Poisson manifold?

In the literature there is some discussion of this question for formal deformation quantizations [GRZ, X1, F]. At a heuristic level one must form a generalized fixed-point algebra of the quantization (as done in [R7] for the special case of actions of  $\mathbb{R}^n$ ). But the technicalities are elusive, and certainly involve the definition of “nice”, which probably involves a suitable notion of “proper action” on non-commutative  $C^*$ -algebras, as discussed in [R2] for exactly these purposes.

Another collection of examples of flabby strict deformation quantizations arises from nilpotent Lie algebras  $\mathfrak{g}$ . Let  $M = \mathfrak{g}^*$  denote the dual vector space of  $\mathfrak{g}$ , with the well-known linear Poisson bracket from  $\mathfrak{g}$  defined by

$$\{f, g\}(\mu) = \langle [df(\mu), dg(\mu)], \mu \rangle.$$

For each  $\hbar \in \mathbb{R}$  define a new Lie bracket  $[\cdot, \cdot]_\hbar$  on  $\mathfrak{g}$  by  $[X, Y]_\hbar = \hbar[X, Y]$ , so that we are contracting  $\mathfrak{g}$  to an Abelian Lie algebra. Let  $G_\hbar$  denote  $\mathfrak{g}$  with the corresponding Lie group structure, and let  $*_\hbar$  denote the corresponding convolution of functions on  $\mathfrak{g}$ . Let  $A = S$  be the algebra of Schwartz functions on  $\mathfrak{g}^*$ , and let  $\hat{\cdot}$  and  $\cdot^\vee$  denote the Fourier transform from  $\mathfrak{g}^*$  to  $\mathfrak{g}$  and its inverse. For  $f, g \in A$  set

$$f \times_\hbar g = (\hat{f} *_\hbar \hat{g})^\vee.$$

With the  $C^*$ -norms coming from the group  $C^*$ -algebras  $C^*(G_\hbar)$  this defines a flabby strict deformation quantization of  $\mathfrak{g}^*$  [R3]. This construction too can be used to construct certain non-compact quantum groups [R4, Ka1, Ka2, Ka3].

**8. Question.** To what extent can more general contractions of Lie groups be seen to give strict deformation quantizations of “non-commutative Poisson algebras” as defined in [BG, N1, N2, X2]?

Brief allusion to this possibility occurs following conjecture 4.29 of [BCH].

But there are interesting strict deformation quantizations which are not flabby. In the example just above, if one lets  $\mathfrak{g}$  be a non-nilpotent exponential solvable Lie group, then convolution does not carry the Schwartz functions into themselves, and so one must take as  $A$  the smooth functions whose Fourier transforms have compact support [R3]. Thus the elements of  $A$  are analytic functions, and so this example is not flabby. In fact, many of the non-flabby examples feel rigid in the sense that analytic functions do. (Does this permit the use of analytic continuation to get more precise information?) This brings us to the most basic questions:

**9. Question.** When does a Poisson manifold admit a strict deformation quantization? In particular, what cohomological obstructions are there to having a strict deformation quantization?

**10. Question.** When does a Poisson manifold admit a flabby strict deformation quantization. In particular, what cohomological obstructions are there? Do there exist Poisson manifolds which admit a strict deformation quantization but do not admit one which is flabby?

My guess is that the answer to this last question is “yes”. But it is striking that up to now there are virtually no negative results for these two questions. It would be interesting to see to what extent the techniques in [WX, X3], which are

concerned with formal deformation quantization, can be adapted to the operator algebra setting, and whether they are of any help in dealing with the above two questions. Very recently Kontsevich [Ko] has shown that every Poisson manifold has a formal deformation quantization. It is quite a challenge to see whether any of his ideas can be adapted to the operator algebra setting.

Many interesting examples of strict deformation quantizations have been constructed in terms of generators and relations, including many quantum groups. (Construction by generators and relations works best for compact spaces, since otherwise the generators tend to give unbounded operators which are technically difficult to work with, though Woronowicz has nevertheless done remarkable things in the situation of unbounded operators.) But most of the examples constructed in this way are not flabby. Nagy [N1, N2] has given a framework for proving that many such constructions give strict deformation quantizations. This framework has been reinforced by Blanchard [Bln]. It would be interesting to know if the results of [BEW] can be brought within this framework.

A more geometrical approach to constructing deformation quantizations is through groupoid  $C^*$ -algebras. See [Sh4, Sh5, Sh6] and the references therein. Again it would be interesting to know what can be said about the flabbiness of the resulting quantizations, as well as their relation to a number of the questions which follow.

It is important to know how a deformation quantization relates to the geometry of the Poisson manifold. A Poisson manifold decomposes into symplectic leaves, and it is natural to expect that these symplectic leaves might correspond somewhat to (two-sided) ideals in the  $C^*$ -algebras of the deformation quantization. If a leaf is not closed, then we would only expect that its closure might correspond to an ideal. (Then, in the nice case in which the leaf is open in its closure, the leaf itself might correspond to an ideal in the quotient algebra.)

**11. Definition.** We say that a deformation quantization of a Poisson manifold (with notation as above) is **leaf-preserving**, or **tangential**, if for each symplectic leaf  $L$  the ideal

$$I_L = \{f \in A : f|_L = 0\}$$

of  $A$  is also a  $*$ -ideal for each of the deformed products and involutions. We also require that the closure  $\bar{L}$  of  $L$  is determined by  $I_L$  in the sense that  $\bar{L}$  consists exactly of every point at which all the functions in  $I_L$  vanish.

The generalized Weyl-Moyal quantization for actions of  $\mathbb{R}^n$  described above is easily seen to be leaf-preserving, using results in [R5]. But the flabby strict deformation quantization described above for nilpotent Lie groups is well-known to not be leaf-preserving most of the time. This appears to be due to the fact that the Fourier transform does not mesh naturally with the group structure. There is evidence that by modifying the Fourier transform to an “adapted Fourier transform”, the deformation quantization for nilpotent Lie groups, and even for exponential groups, can be modified so as to be leaf preserving. See [Be] and references therein. But the full extent to which this can be done successfully at the analytical level does not seem to have been worked out yet. But this raises the more general:

**12. Question.** Does every Poisson manifold which admits a flabby strict deformation quantization, always admit one which is leaf-preserving?

**13. Question.** Are there examples of Poisson manifolds which admit a (non-flabby) strict deformation quantization, but do not admit one which is leaf-preserving?

My guess is that such examples exist. Within the setting of formal deformation quantization this question has been discussed recently in [C2, Ls]. For the geometric analogue see [W2].

In many important situations in which Poisson manifolds arise there is additional structure present which one would like to preserve under quantization. Probably the most common extra structure is a group of symmetries, that is, a group of diffeomorphisms of the manifold which respect the Poisson bracket. One can then ask most of the above questions but with the added requirement that the additional structure be preserved, in a suitable sense. In this setting we have one of the very few “no-go” examples that I know of:

**14. Example** (at the very end of [R1]). Consider the two-sphere with the Poisson bracket from its rotationally invariant symplectic structure. There is no strict deformation quantization of this Poisson manifold which preserves the action of  $SO(3)$ .

It would be interesting to know how common such examples are. That is, we have the following counterpart to Question 9:

**15. Question.** Given a Poisson manifold with action of a group, what are some necessary conditions for the existence of a strict deformation quantization which respects the group action? In particular, what cohomological obstructions are there to such existence?

By means of generators and relations Nagy [N1] has constructed a strict deformation quantization of the disk which respects the action of  $SL(1, 1)$ . This example is not flabby. But the disk with its  $SL(1, 1)$ -invariant Poisson bracket is symplectomorphic to the plane with its standard Poisson bracket, and so admits a strict deformation quantization by the Weyl-Moyal construction, which is flabby, but does not preserve the  $SL(1, 1)$ -action.

**16. Question.** Is there a flabby strict deformation quantization of the disk which preserves the action of  $SL(1, 1)$ ?

I would not be surprised if the answer is “no”.

A related type of additional structure which can be present is the canonical coproduct on the algebra of functions on a Lie group, which one wants to preserve (perhaps in a deformed way) when one is trying to deform the Lie group into a quantum group. For the standard quantum group  $SU_\mu(n)$  of Woronowicz (constructed by generators and relations), it was shown by Sheu [Sh1] for  $n = 2$ , and then by Nagy [N1, N2] for general  $n$ , that they form a strict deformation quantization of  $SU(n)$ . But it is not flabby. Related results about deformations and quantum groups can be found in [Bln].

In this context we have the only other substantial “no-go” example of which I am aware:

**17 Example.** Sheu [Sh2, Sh3] has shown that there is no strict deformation quantization of  $SU(n)$  with its standard Poisson-Lie bracket which simultaneously preserves the comultiplication and is leaf-preserving.

Again, it would be desirable to know how common such examples are. But note in contrast that the strict deformation quantizations of, say,  $SU(n)$  for  $n \geq 3$  as quantum groups which are constructed in [R6] for non-standard Poisson-Lie brackets, are leaf-preserving (and flabby) and preserve the comultiplication.

Given a strict deformation quantization, it is natural to ask whether the deformed  $C^*$ -algebras have the same “algebraic topology” as the original manifold. In particular, do they have isomorphic  $K$ -groups? (In discussing this one probably wants to assume that the set  $I$  over which  $\hbar$  ranges is connected.) Already one sees from the quantum tori that the order structure on  $K_0$  will often be different. For the generalized Weyl-Moyal quantization discussed above it is shown in [R7] that the  $K$ -groups are isomorphic. Very recently Nagy [N3] has developed techniques in  $E$ -theory which deal with the case in which many fiber algebras are non-isomorphic. He applies this to show that the  $K$ -groups are isomorphic in many other cases, in particular for certain quantum groups.

However,  $K$ -groups are not always preserved under strict deformation quantization. For example, let  $M$  be the closed unit interval, with the 0 Poisson bracket on  $C(M)$ . (There are easy variations on this example in which  $M$  is a compact manifold without boundary.) Let  $A$  be the dense subalgebra consisting of the polynomials. Keep the product and involution fixed, and deform only the  $C^*$ -norm on  $A$ , as follows. Let  $N_{\hbar}$  denote the union of the interval  $[\hbar, 1]$  with the sequence  $1/n : n \geq 1$ . Let  $\|\cdot\|_{\hbar}$  be the supremum norm over  $N_{\hbar}$ . Since  $N_{\hbar}$  has an infinite number of components, the completion of  $A$  for each of these norms will have infinitely-generated  $K_0$  group, in contrast to  $C(M)$ . In this example one can also take as  $A$  the functions which are analytic in a neighborhood of the interval  $[0, 1]$ , so as to obtain a “local  $C^*$ -algebra” as defined in [Bla], where the comment in 3.1.6 of [Bla] about partitions of unity is relevant. However, this example is not flabby, and it is possible that flabbiness is useful in connection with the  $K$ -theory of quantization. That is:

**18. Question.** Are the  $K$ -groups of the  $C^*$ -algebra completions of the algebras of any flabby strict deformation quantization all isomorphic?

We remark that the quantum group examples treated by Nagy mentioned above are not flabby but nevertheless the  $K$ -groups are isomorphic. We also remark that the 0 Poisson bracket can have non-commutative strict deformation quantizations. If in any strict deformation quantization of a non-zero Poisson bracket one reparametrizes by replacing  $\hbar$  by  $\hbar^2$  one obtains a strict deformation of the 0 Poisson bracket.

A Poisson manifold has a smooth structure, with its associated deRham cohomology. It would be desirable to have this cohomology preserved under strict deformation quantization. In the non-commutative case the role of the deRham cohomology is played by Connes’ cyclic homology. This requires the use of a dense  $*$ -subalgebra to play the role of the algebra of smooth functions. But in many of the examples constructed by generators and relations, the most evident dense  $*$ -subalgebra feels more like an algebra of polynomials than like the algebra of all smooth functions. In particular, its  $K$ -groups and cyclic homology are unlikely to agree with those of the original manifold. Already for the basic example of Woronowicz’ quantum group  $SU_{\mu}(2)$  the situation is unclear. That is:

**19. Question.** For the quantum group  $SU_{\mu}(2)$  (and also for  $SU_{\mu}(n)$ ), and for any given  $\mu$ , is there a dense  $*$ -subalgebra  $A$  such that:

1. The algebra  $A$  is closed in  $SU_\mu(2)$  under the holomorphic functional calculus, so that it has the same K-groups as  $(SU_\mu(2))$  and so as  $SU(2)$ .
2. The periodic cyclic homology of the algebra  $A$  agrees in the appropriate sense with the deRham cohomology of  $SU(2)$ .
3. The comultiplication of  $SU_\mu(2)$  carries  $A$  into some kind of smooth tensor product  $A \otimes A$ .

It seems to me that only with an affirmative answer can one really say that  $SU_\mu(2)$  has the structure of a non-commutative differentiable manifold with a smooth “group structure”, so that it is not just a quantum group, but in fact a quantum Lie group.

When  $Q$  is a manifold, its cotangent bundle,  $T^*Q$ , carries a canonical symplectic form. When  $Q$  is the configuration space of a classical mechanical system,  $T^*Q$  is its phase space. Thus the problem of quantizing  $T^*Q$  is one of central importance. In many physical situations there is a natural Riemannian metric on  $Q$ . One then expects to need to use this Riemannian metric to obtain a natural quantization of  $T^*Q$ . There is an extensive literature on this subject. Recent discussion appears in [LQ, La1, Om]. A substantial discussion of this will appear in [La3]. But most of the constructions go in the direction of producing a quantization in (at most) the weaker sense of Definition 23 below. Thus there is need for a clearer answer to:

**20. Question.** How often, and in what ways, can a suitable Riemannian metric on  $Q$  be used to obtain a strict deformation quantization of  $T^*Q$ ?

There are recent generalizations of the Weyl quantization to Riemannian symmetric spaces and related situations which appear to have much promise of giving strict deformation quantizations [U1, U2, UU]. But no general proofs have been given yet.

Landsman ([La2, LW] and references therein) has extensively discussed how ideas from induced representation of  $C^*$ -algebras provide the quantum version of the classical technique of symplectic reduction. He indicates in particular how this applies to strict deformation quantization. This provides further examples, but there is much more remaining to be developed in this direction.

There is no uniqueness for strict deformation quantizations, as seen already by considering quantum tori [R8]. But the example of quantum tori suggests that there should be some natural definition of equivalence of strict deformation quantizations which is weak enough to permit the fiber algebras to be non-isomorphic. Thus:

**21 Question.** What are some useful definitions of equivalence for strict deformation quantization?

Even in the absence of uniqueness there are many reasons to believe that various kinds of additional structure on a Poisson manifold may select a natural quantization. At the level of formal deformation quantization of symplectic manifolds this has been realized through the use of symplectic connections by Fedosov. (See the review in [W1].) Speculation that the presence of a Riemannian metric on the Poisson manifold (not just on a configuration space) can serve the same purpose is given in [Kl, KM]. So it is natural to ask:

**22. Question.** What types of additional structure on a Poisson manifold will lead to a canonical choice of a strict deformation quantization? (So in particular, to what extent will the specification of a Riemannian metric do this?)



For many situations it is probably unreasonable to ask for a strict deformation quantization. There is a weakening of the quantization requirements which has permitted many more examples to be constructed. This is often associated with the names Berezin and Toeplitz. But as suggested by Landsman, it is probably best to refer to it simply as “strict quantization”, reserving the names Berezin and Toeplitz for more specific versions of it. We now no longer ask for a deformed product on a Poisson algebra  $A$  of functions.

**23. Definition.** Let  $M$  be a Poisson manifold. By a *strict quantization* of  $M$  we mean a dense  $*$ -subalgebra  $A$  of  $C_\infty(M)$  which is carried into itself by the Poisson bracket, together with a closed subset  $I$  of the real line containing 0 as a non-isolated point, and for each  $\hbar \in I$  a linear map  $T_\hbar$  (usually preserving the involution) of  $A$  into a  $C^*$ -algebra  $\bar{A}_\hbar$  which is generated by the range of  $T_\hbar$ , such that for  $\hbar = 0$  the map  $T_0$  is the canonical inclusion of  $A$  into  $C_\infty(M)$ , and such that

1. The maps  $\hbar \rightarrow T_\hbar(a)$  define the structure of a continuous field of  $C^*$ -algebras on the family  $\{\bar{A}_\hbar\}$ .
2. For  $f, g \in A$  we have

$$\|(T_\hbar(f)T_\hbar(g) - T_\hbar(g)T_\hbar(f))/\hbar - iT_\hbar(\{f, g\})\|_\hbar \rightarrow 0$$

as  $\hbar \rightarrow 0$ .

If each  $T_\hbar$  is injective, we will say that the strict quantization is “faithful”.

The difference from deformation quantization is that for  $f, g \in A$  the product operator  $T_\hbar(f)T_\hbar(g)$  need not be of the form  $T_\hbar(h)$  for some  $h \in A$ , so no product is defined on  $A$ .

Most of the above questions have their counterparts in this weaker setting. We will not spell them out here. However, the precise relation with strict deformation quantization is unclear to me. In particular:

**24. Question.** Is there an example of a Poisson manifold which admits a (faithful?) strict quantization, but does not admit a strict deformation quantization?

More specifically:

**25. Question.** Is there an example of a faithful strict quantization,  $T$ , defined on a dense  $*$ -subalgebra  $A$ , such that it is impossible to find a dense  $*$ -subalgebra  $B \subseteq A$  on which  $T$  is a strict *deformation* quantization, that is, such that for any  $f, g \in B$  and any  $\hbar$  the product operator  $T_\hbar(f)T_\hbar(g)$  is of the form  $T_\hbar(h)$  for some  $h \in B$ , where  $h$  depends on  $\hbar$ ?

When  $T_\hbar$  is injective, it is appropriate to consider  $\bar{A}_\hbar$  to be a quantum version of the Poisson manifold  $M$ . It then becomes very interesting to determine the properties of this  $C^*$ -algebra (e.g. its structure and its algebraic topology). A certain amount is known in some cases, as referenced in [R8]. Further information and some interesting conjectures about the structure of other examples can be found in [BLR]. It would be interesting to know if the results of [C1] and its predecessors could also be brought within this setting.

In many of the examples which have been constructed, the  $T_\hbar$ ’s are suitable Toeplitz operators, that is, they are compressions to suitable subspaces of a representation of  $C_\infty(M)$ . See references in [R8, AG], and the more recent papers [BMS, Bo, Sc] and their references. We will refer to such a quantization as a “strict Toeplitz quantization”. In this case the quantization is clearly “positive”

in the sense that positive functions are taken to positive operators. But for the Weyl-Moyal-type quantizations positivity usually fails.

**26. Question.** Is it impossible for a strict deformation quantization to be positive?

**27. Question.** If a Poisson manifold admits a strict quantization, does it always admit one which is positive?

For many of the examples which have been constructed, especially for compact  $M$ , the  $C^*$ -algebras  $\bar{A}_\hbar$  are finite-dimensional (see [BMS, Bo, Sc] and references therein), so that it does not seem appropriate to consider them to be quantum versions of  $M$ . Thus it would be very interesting to have alternative constructions which produce  $T_\hbar$ 's which are injective.

**28. Question.** If a Poisson manifold admits a strict quantization, does it always admit one which is faithful?

There is one important situation in which recently this has been successfully shown to be true. There has been much interest in the construction of quantum versions of compact Riemann surfaces of genus  $\geq 2$ . Most of the attempted constructions run into this problem of producing finite dimensional Hilbert spaces. However, Klimek and Lesniewski [KL] found a way around this difficulty by using a cleverly-chosen non-compact covering of the Riemann surface with covering group  $\mathbb{Z}$ , constructing a Toeplitz quantization of this covering surface respecting the action of  $\mathbb{Z}$ , and showing that there are plenty of  $\mathbb{Z}$ -invariant operators, which then give the quantization of the compact surface. The exploration of the properties of these quantum Riemann surfaces should be an interesting adventure in the years ahead. (I have learned recently that T. Natsume and R. Nest have developed a somewhat different approach to constructing quantum Riemann surfaces. It starts with the action of the fundamental group of the Riemann surface on the quantization of the disk by Toeplitz operators. They then form the cross product algebra, and then cut down by carefully chosen projections. They are able to obtain substantial information about the properties of the resulting algebras.)

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